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Computation of Gauss-type quadrature formulas with some preassigned nodes†

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Abstract

When dealing with the approximate calculation of weighted integrals over a finite interval $[a, b]$, Gauss-type quadrature rules with one or two prescribed nodes at the end points $\{a, b\}$ are well known and commonly referred as Gauss-Radau and Gauss-Lobatto formulas respectively. In this regard, efficient algorithms involving the solution of an eigenvalue problem for certain tri-diagonal (Jacobi) matrices are available for their computation. In this work a further step will be given by adding to the above quadratures an extra fixed node in (a, b) and providing similar efficient algorithms for their computation. This will be done by passing to the unit circle and taking advantage of the so-called Szegő-Lobatto quadrature rules recently introduced in [27] and [6].

Keywords: Szegő-type quadrature formulas, Gauss-type quadrature formulas, para-orthogonal polynomials, Jacobi matrices.

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§1. Introduction and preliminary results

Given the weighted integral $I_\sigma(f) = \int_a^b f(x)\sigma(x)dx$, $-\infty \le a < b \le +\infty$, σ being a weight function on $[a, b]$, by an n-point quadrature formula we mean an expression like,

$$I_n(f) = \sum_{j=1}^{n} A_j f(x_j), \quad x_j \neq x_k \text{ if } j \neq k \text{ and } \{x_j\}_{j=1}^n \subset [a, b],$$

where the nodes $\{x_j\}_{j=1}^n$ and the coefficients or weights $\{A_j\}_{j=1}^n$ should be conveniently chosen so that $I_n(f)$ provides an approximation to $I_\sigma(f)$. By assuming that the integrals (moments) $c_k = \int_a^b x^k\sigma(x)dx$, $k = 0, 1, \ldots$ exist and can be easily computed and taking into account the density of the polynomials in the class of the continuous functions on $[a, b]$, it seems advisable to choose $\{x_j\}_{j=1}^n$ and $\{A_j\}_{j=1}^n$ so that $I_n(P) = I_\sigma(P)$ for any polynomial P of as high degree N as possible; in the sequel, $\mathcal{P}_k$ denotes the space of polynomials of degree less than or equal to k and $\mathcal{P}$ the space of all polynomials i.e., $\mathcal{P} = \cup_{k=0}^\infty \mathcal{P}_k$. It is known that $n - 1 \le N \le 2n - 1$ and that the case $N = 2n - 1$ gives rise to the so-called Gauss-Christoffel, Gauss or Gaussian formulas with the highest algebraic degree of precision and characterized by the following (see e.g. [30, pp. 101-103] and [38, Theorem 3.4.2]),

Theorem 1.1. A quadrature rule $I_n(f) = \sum_{j=1}^n A_j f(x_j)$ coincides with the n-point Gauss formula for σ or $I_\sigma(f)$, if and only if,

1. The nodes $\{x_j\}_{j=1}^n$ are the zeros of any orthogonal polynomial of degree n with respect to σ.

2. $A_j = \left[\sum_{k=0}^{n-1} Q_k^2(x_j)\right]^{-1} > 0$ for all $j = 1, \ldots, n$ (Christoffel numbers of order n), $\{Q_k\}_{k=0}^\infty$ being the sequence of orthonormal polynomials with respect to σ.

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Gaussian formulas exhibit the following interesting features:

1. They are *optimal* in the sense that for each $n \geq 1$, $I_n(P) = I_\sigma(P)$, for all $P \in \mathcal{P}_{2n-1}$ and there exists $Q \in \mathcal{P}_{2n}$ such that $I_\sigma(Q) \neq I_n(Q)$.

2. Their coefficients $\{A_j\}_{j=1}^n = \{A_j^{(n)}\}_{j=1}^n$ are positive which automatically implies stability i.e., for all $n = 1, 2, \ldots$ there exists a positive constant M such that $\sum_{j=1}^n |A_j^{(n)}| \leq M$.

3. Convergence is guaranteed in the class of bounded functions f on $[a, b]$ such that $f\sigma$ is integrable on $[a, b]$.

4. An efficient computation of the nodes and weights can be carried out in terms of an eigenvalue problem involving certain tri-diagonal (Jacobi) matrices associated with the sequence $\{Q_k\}_{k=0}^\infty$.

In the Gaussian formula no freedom is left to fix some nodes in advance. However, in some applied problems, specially concerned with the numerical solution of differential and integral equations, it is required to construct quadrature formulas with some of the nodes given beforehand so that the remaining nodes and weights could be chosen by satisfying similar features to the Gaussian ones, that is, positive weights and exactly integrating polynomials up to as high degree as possible. Typical examples are the well known Gauss-Radau and Gauss-Lobatto formulas (which we will refer to below) as well as the integration rules introduced by Kronrod [29] and giving rise to the so called Gauss-Kronrod formulas. These are rules of the form $I_{2n+1}(f) = \sum_{i=1}^n A_i f(\xi_i) + \sum_{j=1}^{n+1} B_j f(x_j)$, where the nodes $\{\xi_i\}_{i=1}^n$ are the zeros of the polynomial of degree n orthogonal on (a, b) with respect to σ while the nodes $\{x_j\}_{j=1}^{n+1}$ and the weights $\{A_i\}_{i=1}^n \cup \{B_j\}_{j=1}^{n+1}$ should be optimally chosen in the sense that it holds that $I_{2n+1}(f) = I_\sigma(f)$, for all $f \in \mathcal{P}_{3n+1}$. Furthermore, the nodes $\{x_j\}_{j=1}^{n+1}$ must be mutually distinct and lie in $[a, b]$. In general, the existence of these rules can not be assured, see [7], [15] and [31].

Gauss-Kronrod formulas are a particular case of a more general situation where we will try to construct rules of the type

$$I_{n+m}(f) = \sum_{i=1}^m A_i f(a_i) + \sum_{i=1}^n B_j f(x_j), \tag{1.1}$$

with the nodes $\{a_i\}_{i=1}^m$ fixed in advance. Since (1.1) depends on $2n + m$ parameters, we could try to choose them so that the algebraic degree of precision is maximal i.e., $I_{n+m}(P) = I_\sigma(P)$, for all $P \in \mathcal{P}_{2n+m-1}$. For this purpose, let us introduce the polynomials,

$$E_m(x) = (x - a_1) \cdots (x - a_m) \quad \text{and} \quad p_n(x) = (x - x_1) \cdots (x - x_n). \tag{1.2}$$

Then, it is not difficult to prove the following (see e.g. [13, pp. 109-112], [30, pp. 101-102], [34, Theorem 1] and [35]),

Theorem 1.2. The algebraic degree of precision of formula (1.1) is equal to $2n + m - 1$, if and only if, it is of interpolatory type in $\mathcal{P}_{m+n-1}$ i.e., it is exact in $\mathcal{P}_{n+m-1}$, and the polynomial p_n in (1.2) is orthogonal on (a, b) with respect to σE_m to every polynomial of degree $n - 1$ i.e.,

$$\int_a^b p_n(x) x^k E_m(x) \sigma(x) dx = 0, \; k = 0, 1, \ldots, n - 1. \tag{1.3}$$

In view of Theorem 1.2 it is clear that the construction of rules of type (1.1) with algebraic degree of precision $2n + m - 1$ reduces to finding the polynomial $p_n(x)$ satisfying (1.3). Furthermore, the roots of $p_n(x)$ must be real, distinct and lie on $[a, b]$. They must be also distinct from the fixed nodes $\{a_i\}_{i=1}^m$. This general situation clearly leads to the study of orthogonal polynomials with respect to signed functions on the interval $[a, b]$.

Certainly, when the nodes $\{a_i\}_{i=1}^m$ are chosen outside (a, b), a rule like (1.1) can be always constructed. This is the case when dealing with a finite interval $[a, b]$, say $[-1, 1]$, and considering $m = 1, 2$ with $a_i \in \{\pm 1\}$ and yielding the well known Gauss-Radau ($m = 1$) and Gauss-Lobatto ($m = 2$) quadrature rules, as summarized in the following

Theorem 1.3. Given $r, s \in \{0, 1\}$, consider the n-point rule $I_n^{r,s}(f) = rA_n^+ f(1) + sA_n^- f(-1) + \sum_{j=1}^{n-r-s} A_j^{r,s} f(x_j^{r,s})$. Then, $I_n^{r,s}(P) = I_\sigma(P)$, for all $P \in \mathcal{P}_{2n-1-r-s}$, if and only if,

1. $I_n^{r,s}$ is of interpolatory type in $\mathcal{P}_{n-1}$.

2. The nodes $\{x_j^{r,s}\}_{j=1}^{n-r-s}$ are the zeros of $p_n^{r,s}(x)$, the orthogonal polynomial of degree $n-r-s$ with respect to $\sigma_{r,s}(x) = (1-x)^r(1+x)^s\sigma(x)$, $x \in [-1,1]$. Furthermore, the weights are positive and it holds that
$$A_j^{r,s} = \frac{\tilde{A}_j^{r,s}}{(1-x_j^{r,s})^r(1+x_j^{r,s})^s}, \quad j = 1,\dots,n-r-s, \tag{1.4}$$
$\{\tilde{A}_j^{r,s}\}_{j=1}^{n-r-s}$ being the Christoffel numbers for $\sigma_{r,s}$ of order $n-r-s$.

Remark 1.4. For the proof of this well known result see e.g. [13]. The relation (1.4) can be deduced from $A_j^{r,s} = \int_a^b \frac{p_n^{r,s}(x)}{\left(x-x_j^{r,s}\right)(p_n^{r,s})'(x_j)}\sigma_{r,s}(x)dx$.

Gauss-Radau and Gauss-Lobatto formulas along with Gaussian rules are sometime called *Gauss-type formulas* so that its computation can be efficiently carried out by means of the solution of an eigenvalue problem involving certain tri-diagonal (Jacobi) matrices arising from the three-term recurrence relation satisfied by any sequence of orthogonal polynomials, see e.g. [16]-[17], among others.

In this work, we will go a step further by allowing to add to the nodes of the Gauss-type formulas an extra fixed point in $(a,b) = (-1,1)$ and giving rise to the following problem: given $x_\alpha \in (-1,1)$, $r,s \in \{0,1\}$ and $n > 1+r+s$, find positive weights $A_+^{r,s}, A_-^{r,s}, A_\alpha^{r,s}$ and $\{A_j^{r,s}\}_{j=1}^{n-r-s-1}$ along with distinct nodes $\{x_j^{r,s}\}_{j=1}^{n-r-s-1} \subset (-1,1)$ such that,
$$\begin{array}{rcl} I_n^{r,s}(f) & = & rA_+^{r,s}f(1) + sA_-^{r,s}f(-1) + A_\alpha^{r,s}f(x_\alpha) + \sum_{j=1}^{n-1-r-s} A_j^{r,s}f(x_j^{r,s}) \\ & = & I_\sigma(f), \quad \text{for all } f \in \mathcal{P}_{2(n-1)-r-s}. \end{array} \tag{1.5}$$
The particular case $r = s = 0$ and $\sigma(x) \equiv 1$ has been recently studied in [22] by showing how these quadrature rules can be used to construct efficient Runge-Kutta methods for the numerical solution of stiff problems and algebraic differential equations. On the other hand, in a recent paper A. Bultheel et al. [4] have dealt with the general situation by means of the theory of orthogonal polynomials with respect to σ on (a,b).

In this paper a more unified and simpler approach to the solution of (1.5) will be given by passing from the interval $[-1,1]$ to the unit circle through the Joukowski[1] transformation $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$ conformally mapping the interior of the unit circle $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto the exterior of $[-1,1]$. Thus, the weight function σ on $[-1,1]$ produces a new weight function $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ or equivalently on any interval of length 2π, which will be taken as $[-\pi,\pi]$.

In this respect, it will be shown that when considering $r = s$, either $I_n^{0,0}(f)$ or $I_{n+1}^{1,1}(f)$ always exists but never both simultaneously. On the other hand, the case $r \neq s$ has always a solution too, either $r = 1$ and $s = 0$ or $r = 0$ and $s = 1$ but not both simultaneously. Furthermore, in all of the cases when existence is assured, the weights in (1.5) are positive. Finally, it should be remarked that in case there exists a rule $I_n^{r,s}(f)$ of type (1.5), it is unique. Indeed, setting $Q_n^{r,s}(x) = \prod_{j=1}^{n-1-r-s}(x - x_j^{r,s}) \in \mathcal{P}_{n-1-r-s}$, then from Theorem 1.2, it follows that
$$\int_{-1}^{+1} x^k Q_n^{r,s}(x)(1-x)^r(1+x)^s(x-x_\alpha)\sigma(x)dx = 0, \quad k = 0,1,\dots,n-2-r-s.$$
Setting again $\sigma_{r,s}(x) = (1-x)^r(1+x)^s\sigma(x)$ one sees that the polynomial $(x-x_\alpha)Q_n^{r,s}(x) \in \mathcal{P}_{n-r-s}$ is orthogonal to any polynomial in $\mathcal{P}_{n-2-r-s}$ with respect to $\sigma_{r,s}$. Now, assume that $\{P_k^{r,s}(x)\}_{k=0}^{\infty}$ is the sequence of monic orthogonal polynomials with respect to $\sigma_{r,s}$. Then, $(x-x_\alpha)Q_n^{r,s}(x) = P_{n-r-s}^{r,s}(x) + \lambda P_{n-1-r-s}^{r,s}(x)$ for some $\lambda \in \mathbb{R}$. Observe the restriction $P_{n-r-s}^{r,s}(x_\alpha) \neq 0$; otherwise, x_α is a common zero of $P_{n-r-s}^{r,s}(x)$ and $P_{n-1-r-s}^{r,s}(x)$, which is a contradiction. Then, $\lambda = -\frac{P_{n-r-s}^{r,s}(x_\alpha)}{P_{n-1-r-s}^{r,s}(x_\alpha)}$ and the nodes $\{x_j^{r,s}\}_{j=1}^{n-1-r-s}$ are uniquely determined, yielding the unicity of formula (1.5).

The paper has been organized as follows. In Section 2 we summarize briefly the construction of Szegő-type (Szegő, Szegő-Radau and Szegő-Lobatto) quadrature rules, emphasizing the situation when the corresponding weight function on the unit circle is symmetric. From these results we will

[1] Also called the Joukowsky or Zhukovsky transform.

characterize in Section 3 the construction of a Gauss-type (Gauss, Gauss-Radau and Gauss-Lobatto) quadrature formula with an additional preassigned node inside the interval of integration. This will be done by passing to the unit circle and in a simpler approach we will recover the recent results deduced in [4]. In Section 4 we present an algorithm for the computation of such rules involving the solution of an eigenvalue problem for certain Jacobi matrices whereas in Section 5 we finally present some illustrative numerical examples involving Bernstein-Szegő polynomials.

§2. Periodic integrands: Szegő-type formulas

In this section we will briefly survey the construction of n-point quadrature rules in order to approximate the integral

$$I_\omega(g) = \int_{-\pi}^{\pi} g(\theta)\omega(\theta)d\theta, \tag{2.1}$$

where both g and ω are 2π-periodic functions and ω is a weight function on any interval of length 2π.

For our purposes, we will start from an introductory example concerning elliptic integrals of the form (see [14])

$$G(a,b) = \int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}, \quad (\omega(\theta)\equiv 1), \tag{2.2}$$

with a and b fixed. $G(a,b)$ could be estimated by means of the Gauss-Legendre quadrature rule. On the other hand, $G(a,b)$ can also be approximated by the Trapezoidal rule. The absolute errors are displayed on Table 1 by showing that the Trapezoidal rule provides better results than the Gaussian one. An explanation can be deduced from Example 2.2 given below.

a	b	Gauss-Legendre	Trapezoidal
2.0	3.0	$4.16233E-09$	$4.43762E-12$
0.3	0.2	$4.16233E-08$	$4.43762E-11$
4.0	5.0	$2.11498E-11$	$6.20000E-17$
1.0	2.0	$6.40799E-07$	$7.55989E-08$
0.9	1.0	$4.77908E-12$	$1.00000E-17$
0.4	0.8	$1.60200E-06$	$1.88997E-07$
0.3	0.6	$2.13600E-06$	$2.51996E-07$
0.9	1.6	$1.83510E-07$	$3.86458E-09$
0.9	1.8	$7.11999E-07$	$8.39988E-08$

Table 1: A comparison of the absolute errors in the approximation of the elliptic integrals (2.2) by means of the Gauss-Legendre and Trapezoidal rules.

In general, the integral (2.1) will be approximately calculated by means of,

$$I_n^\omega(g) = \sum_{j=1}^{n} \lambda_j f(\theta_j), \quad \{\theta_j\}_{j=1}^n \subset [-\pi,\pi) \text{ and } \theta_j \neq \theta_k \text{ if } j \neq k,$$

by requiring that $I_n^\omega(T) = I_\omega(T)$ for any trigonometric polynomial T with as high degree N as possible. In this respect, it is known (see [37]) that $N \le n-1$, so that the *optimal* case $N = n-1$ gives rise to the so-called quadrature formulas with the maximum trigonometric degree of precision. Such formulas are characterized in terms of bi-orthogonal systems of trigonometric polynomials associated with ω. For further details, see also [11]. Alternatively, by passing to the unit circle one can write, $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$, so that if we take into account that any trigonometric polynomial of degree N, $T(\theta) = a_0 + \sum_{j=1}^{N}(a_j\cos j\theta + b_j \sin j\theta)$, can be expressed as $T(\theta) = L(e^{i\theta})$ with $L(z) = \sum_{j=-N}^{N}\alpha_j z^j$, $\alpha_j \in \mathbb{C}$, one tries to find,

$$I_n^\omega(g) = \sum_{j=1}^{n} \lambda_j g(z_j), \quad \{z_j\}_{j=1}^n \subset \mathbb{T},\ z_j \neq z_k \text{ if } j \neq k, \tag{2.3}$$

such that

$$I_n^\omega(L) = I_\omega(L), \quad \text{for all } L \in \Lambda_{-(n-1),n-1}. \tag{2.4}$$

Here, given p and q integers with $p \leq q$, $\Lambda_{p,q} = \text{span}\{z^k : p \leq k \leq q\}$ (Laurent polynomials) and $\Lambda = \text{span}\{z^k : k \in \mathbb{Z}\}$ (the space of all Laurent polynomials).

The construction of the formula (2.3) satisfying (2.4) can be done by means of Szegő polynomials. Indeed, if we denote by $\{\rho_n(z)\}_{n=0}^{\infty}$ the family of monic orthogonal (Szegő) polynomials for ω (see [36] and [38, Chapter 11]) it holds (see [23] and [28]),

Theorem 2.1. Set $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$ and let $I_n^\omega(g) = \sum_{j=1}^n \lambda_j g(z_j)$ with $z_j \in \mathbb{T}$ for all $j = 1, \dots, n$ and $z_j \neq z_k$ if $j \neq k$. Then $I_n^\omega(g) = I_\omega(g)$, for all $g \in \Lambda_{-(n-1),n-1}$, if and only if,

1. $\{z_j\}_{j=1}^n$ are the zeros of $B_n(z, \tau_n) = z\rho_{n-1}(z) + \tau_n \rho_{n-1}^*(z)$ for some $\tau_n \in \mathbb{T}$, where $\rho_{n-1}^*(z) = z^{n-1}\overline{\rho_{n-1}(1/\overline{z})}$,
2. $\lambda_j = \left(\sum_{k=0}^{n-1} |\varphi_k(z_j)|^2 \right)^{-1} > 0$, for all $j = 1, \dots, n$, where $\{\varphi_n(z)\}_{n=0}^{\infty}$ denotes the corresponding orthonormal family.

$I_n^\omega(g)$ as given in Theorem 2.1 is called an *n-point Szegő quadrature rule* (see [28]) and represents the analog on $\mathbb{T}$ of the Gaussian formulas. These quadratures are also *optimal* in the sense that there can not exist an n-point rule with nodes on $\mathbb{T}$ integrating Laurent polynomials either in $\Lambda_{-n,n-1}$ or in $\Lambda_{-(n-1),n}$ exactly. However, as mentioned in [39, Section 12], it can be proved that an n-point Szegő formula is exact in $\mathcal{L}_n$ such that $\dim(\mathcal{L}_n) = 2n$ and $\Lambda_{-(n-1),n-1} \subset \mathcal{L}_n \subset \Lambda_{-n,n}$ (see also [33]).

Example 2.2. Suppose $\omega(\theta) \equiv 1$ and let g be a 2π-periodic function. Now from Theorem 2.1 and since $\rho_n(z) = z^n$ (see e.g. [38, Chapter 11]) for all $n = 1, 2, \dots$ it follows that $B_n(z, \tau_n) = z^n + \tau_n$. Hence, the nodes $\{z_j\}_{j=1}^n$ are the n-th roots of $-\tau_n \in \mathbb{T}$. On the other hand, $\|\rho_n\|_\omega^2 = \int_{\mathbb{T}} |\rho_n(z)|^2 dz = 2\pi$ for all $n = 0, 1, \dots$ yielding $\lambda_j = \frac{2\pi}{n}$. Thus $I(g) = \int_{-\pi}^{\pi} g(\theta)d\theta \approx I_n(g) = \frac{2\pi}{n}\sum_{j=1}^n g(\theta_j)$ with $\{\theta_j\}_{j=1}^n$ uniformly distributed on $[-\pi, \pi]$ i.e., $I_n^\omega(g)$ coincides with the n-point Trapezoidal rule applied to $I(g)$. This may explain the numerical results produced by the Trapezoidal rule for the integral $G(a, b)$ in Table 1, since in this case the integrand is clearly a 2π-periodic function.

Now, from the recurrence relation for $\{\rho_n\}_{k=0}^{\infty}$ (see [20], [26], [36, Theorem 1.5.2] or [38, Theorem 11.4.2]),

$$\begin{pmatrix} \rho_{n+1}(z) \\ \rho_{n+1}^*(z) \end{pmatrix} = \begin{pmatrix} z & \delta_{n+1} \\ \overline{\delta_{n+1}}z & 1 \end{pmatrix} \begin{pmatrix} \rho_n(z) \\ \rho_n^*(z) \end{pmatrix}, \quad n = 0, 1, \dots, \tag{2.5}$$

with $\rho_0(z) = \rho_0^*(z) = 1$, $\delta_0 = 1$ and $\delta_n = \rho_n(0) \in \mathbb{D}$ for all $n \geq 1$ (Verblunsky parameters[2]), one sees that in order to generate an n-point Szegő formula, it essentially requires the parameters $\delta_0, \delta_1, \dots, \delta_{n-1}$, along with some τ_n which is freely taken on $\mathbb{T}$. In this respect, for $\tau \in \mathbb{C}$ define the matrix

$$H_n(\tau) = D_n^{-1/2} \begin{pmatrix} -\delta_1 & -\delta_2 & \cdots & -\delta_{n-1} & -\tau \\ \sigma_1^2 & -\overline{\delta_1}\delta_2 & \cdots & -\overline{\delta_1}\delta_{n-1} & -\overline{\delta_1}\tau \\ 0 & \sigma_2^2 & \cdots & \overline{\delta_2}\delta_{n-1} & -\overline{\delta_2}\tau \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{n-1}^2 & -\overline{\delta_{n-1}}\tau \end{pmatrix} D_n^{1/2}, \tag{2.6}$$

where $\sigma_k = \sqrt{1 - |\delta_k|^2} \in (0, 1]$, $k = 1, 2, \dots n$ and $D_n = \text{diag}[\gamma_0, \dots, \gamma_{n-1}] \in \mathbb{R}^{n \times n}$ with $\gamma_0 = 1$ and $\gamma_k = \gamma_{k-1}\sigma_k^2 > 0$, $k = 1, \dots, n-1$. Under these conditions one has (see [24]-[25] and also [8]),

Theorem 2.3. $H_n(\tau)$ given in (2.6) is an unreduced unitary upper Hessenberg matrix for all $\tau \in \mathbb{T}$, so that its eigenvalues $\{z_j\}_{j=1}^n$ which are distinct and of unit magnitude are the zeros of $B_n(z, \tilde{\tau})$ with $\tilde{\tau} = \frac{\tau + \delta_n}{1 + \tau\overline{\delta_n}} \in \mathbb{T}$ or equivalently, the nodes of the n-point Szegő formula for the parameter $\tilde{\tau}$. Furthermore, the square of the first component of the eigenvector of unit length associated with z_j yields the weight λ_j under the assumption that $\mu_0 = \int_{-\pi}^{\pi} \omega(\theta)d\theta = 1$.

[2]There are at least four other terms: Szegő, reflection, Schur and Geronimus parameters, see [36, Chapter 1.5].

Unlike the Gaussian formulas, an n-point Szegő formula is not uniquely determined because of the parameter $\tau_n \in \mathbb{T}$. Thus, given $z_\alpha \in \mathbb{T}$ one can take $\tilde{\tau}_n = -\frac{z_\alpha \rho_{n-1}(z_\alpha)}{\rho^*_{n-1}(z_\alpha)} \in \mathbb{T}$ (observe that $\rho^*_{n-1}(z_\alpha) \neq 0$) so that $B_n(z_\alpha, \tilde{\tau}_n) = 0$. Hence, for this parameter $\tilde{\tau}_n$ an n-point Szegő formula with a prescribed node can be produced and called an n-point Szegő-Radau quadrature rule with a given fixed node at z_α. On the other hand, suppose $z_\alpha, z_\beta \in \mathbb{T}$ such that $z_\alpha \neq z_\beta$ and take $n > 2$. Then (see [27] and [6]) there exist complex numbers $\tilde{\delta}_{n-1} \in \mathbb{D}$ and $\tilde{\tau}_n \in \mathbb{T}$ such that z_α and z_β are zeros of

$$\tilde{B}_n(z, \tilde{\tau}_n) = z\tilde{\rho}_{n-1}(z) + \tilde{\tau}_n \tilde{\rho}^*_{n-1}(z) \text{ where } \tilde{\rho}_{n-1}(z) = z\rho_{n-2}(z) + \tilde{\delta}_{n-1}\rho^*_{n-2}(z). \tag{2.7}$$

Furthermore, $\tilde{B}_n(z, \tilde{\tau}_n)$ has exactly n distinct zeros on $\mathbb{T}$ so that if we denote them by $\{z_\alpha, z_\beta\} \cup \{z_j\}_{j=1}^{n-2}$, there exist positive numbers A, B and λ_j, $j = 1, \ldots, n-2$ such that

$$\tilde{I}^\omega_n(g) = Ag(z_\alpha) + Bg(z_\beta) + \sum_{j=1}^{n-2} \lambda_j g(z_j) = I_\omega(g), \quad \text{for all } g \in \Lambda_{-(n-2),n-2}. \tag{2.8}$$

Formula (2.8) is called an n-point Szegő-Lobatto rule for $I_\omega(g)$ with prescribed nodes at z_α and z_β. Szegő, Szegő-Radau and Szegő-Lobatto will be sometimes referred to as Szegő-type quadrature rules.

Let us nextly analyze the situation when $\omega(\theta)$ is symmetric i.e., $\omega(-\theta) = \omega(\theta)$, $\theta \in [-\pi, \pi]$. Setting $\mu_k = \int_{-\pi}^{\pi} e^{-ik\theta}\omega(\theta)d\theta$ (trigonometric moments) for all $k = 0, \pm 1, \pm 2, \ldots$ and considering the sequence $\{\delta_k\}_{k=0}^\infty$ of Verblunsky parameters, it follows (see e.g. [36])

Lemma 2.4. The following statements are all equivalent:

1. ω is a symmetric weight function on $[-\pi, \pi]$.
2. The Toeplitz matrices associated with ω are symmetric, i.e. $\mu_{-k} = \mu_k$ for all $k \in \mathbb{Z}$.
3. The trigonometric moments are real, i.e. $\mu_k \in \mathbb{R}$ for all $k \in \mathbb{Z}$.
4. The Verblunsky parameters δ_k lie in $(-1, 1)$ for all $k \geq 1$.

Thus, given a symmetric weight function ω, an n-point Szegő formula $I^\omega_n(g) = \sum_{j=1}^n \lambda_j g(z_j)$ is said to be *symmetric* if its nodes $\{z_j\}_{j=1}^n$ are either real or appear in conjugate pairs. The following can be easily proved (see [9]).

Theorem 2.5. Let ω be a symmetric weight function on $[-\pi, \pi]$ and let $I^\omega_n(g) = \sum_{j=1}^n \lambda_j g(z_j)$ be an n-point Szegő formula for $I_\omega(g)$, generated by $B_n(z, \tau_n)$. Then,

1. $I^\omega_n(g)$ is symmetric, if and only if, $\tau_n \in \{\pm 1\}$.
2. Suppose that $z_j = \overline{z}_k$ for some j and k, $1 \leq j, k \leq n$, then $\lambda_j = \lambda_k$.

From Theorem 2.5, we see that when dealing with symmetric rules, computation of their nodes and weights essentially reduces by one half. For further details see the forthcoming paper [3].

Finally, if the integrand and the weight function are symmetric, the choice of complex conjugate Szegő nodes with equal weights seems to be natural. Let $z_\alpha, \overline{z}_\alpha \in \mathbb{T}\backslash\{\pm 1\}$ be fixed in advance. As already seen, for $n > 2$ there exist positive numbers A, B and λ_j, $j = 1, \ldots, n-2$ along with $n-2$ distinct nodes $\{z_j\}_{j=1}^{n-2} \subset \mathbb{T}$ ($z_j \notin \{z_\alpha, \overline{z}_\alpha\}$) such that (2.8) holds with $z_\beta = \overline{z}_\alpha$. Even more, the following results holds. Here, it should be recalled that in [27] existence is assured but not unicity.

Theorem 2.6. Let ω be a symmetric weight function and take $z_\alpha \in \mathbb{T}\backslash\{\pm 1\}$. Then, for $n > 2$ there exists a unique n-point symmetric Szegő-Lobatto quadrature formula for $I_\omega(f)$ with prescribed nodes at z_α and $\overline{z}_\alpha$.

Proof.- For $z \in \mathbb{T}$, define

$$a = a(z, n) = \frac{z^{n-2}\overline{\rho_{n-1}(z)}}{\rho_{n-1}(z)} \in \mathbb{T}. \tag{2.9}$$

For $z, \tau \in \mathbb{T}$ it clearly follows that $B_n(z, \tau) = z\rho_{n-1}(z) + \tau\rho^*_{n-1}(z) = 0$, if and only if, $\tau = -\overline{a}$. Thus, suppose first the fixed node z_α is a zero of $B_n(z, \pm 1)$, yielding $a = a(z_\alpha, n) = \mp 1$ or equivalently,

$$\Im(a) = 0. \tag{2.10}$$

Since both z_α and $\overline{z}_\alpha$ are nodes of the corresponding n-point Szegő formula for $\tau = \pm 1$, the proof is immediately concluded. Secondly, suppose $B_n(z_\alpha, \pm 1) \neq 0$. Now, as shown in [27, Section 3], the construction of an n-point Szegő-Lobatto formula $I_n^\omega(g)$ as given by (2.8) depends on the parameters $\tilde{\delta}_{n-1} \in \mathbb{D}$ and $\tilde{\tau}_n \in \mathbb{T}$ appearing in (2.7). Furthermore, when ω is symmetric, $\tilde{\delta}_{n-1}$ can be taken real i.e., $\tilde{\delta}_{n-1} \in (-1,1)$ and then $\tilde{\tau}_n \in \{\pm 1\}$. More precisely,

$$\tilde{\tau}_n = -\text{sign}\left(\frac{\Im(z_\alpha)}{\Im(a)}\right). \tag{2.11}$$

Thus, the coefficients of the polynomial $\tilde{B}_n(z, \tilde{\tau}_n)$ in (2.7) are real and therefore its zeros are real or appear in complex conjugate pairs on $\mathbb{T}$, so that the symmetric character of the rule is guaranteed. Therefore, take into account that the n-point Szegő-Lobatto formula $I_n^\omega(g)$ is actually an n-point Szegő rule for $I_{\tilde{\omega}}(g)$, $\tilde{\omega}$ being a new symmetric weight function whose n first Verblunsky parameters are $1, \delta_1, \ldots, \delta_{n-2}$ and $\tilde{\delta}_{n-1}$. Hence, it remains to prove the unicity of $\tilde{\delta}_{n-1} \in (-1,1)$. The parameter $\tilde{\delta}_{n-1}$ satisfies the equation, (see [27, Equation (27)] and take into account that $\tilde{\delta}_{n-1} \in (-1,1)$)

$$x^2 - 2cx - (1 + 2c\Re(a)) = 0, \quad \text{with } c = -\frac{\Im(z_\alpha)}{\Im(az_\alpha)} \tag{2.12}$$

and $a \in \mathbb{T}$ given by (2.9). Since $z_\alpha \neq \pm 1$, then $c \neq 0$.

Let x_1 and x_2 be the roots of (2.12); it is known that both are real and at least one of them lies on $(-1,1)$. Thus, we only need to prove that either $|x_1| \geq 1$ or $|x_2| \geq 1$. From (2.12) it follows

$$x_1 = c\left(1 + \sqrt{1 + \frac{1}{c}\left(\frac{1}{c} + 2\Re(a)\right)}\right) \quad \text{and} \quad x_2 = c\left(1 - \sqrt{1 + \frac{1}{c}\left(\frac{1}{c} + 2\Re(a)\right)}\right).$$

Obviously if $|c| \geq 1$ then $|x_1| \geq 1$. So, let us analyze the case $|c| < 1$. Suppose first that $c < 0$. Then $|c|^{-1} = -c^{-1} > 1$. Since $a \in \mathbb{T}$,

$$\Re(a) \leq 1 \Leftrightarrow \frac{2\Re(a)}{c} \geq \frac{2}{c} \Leftrightarrow \frac{2\Re(a)}{c} + \frac{1}{c^2} \geq \frac{2}{c} + \frac{1}{c^2} \Leftrightarrow 1 + \frac{1}{c}\left(\frac{1}{c} + 2\Re(a)\right) \geq \left(1 + \frac{1}{c}\right)^2.$$

Thus,

$$\sqrt{1 + \frac{1}{c}\left(\frac{1}{c} + 2\Re(a)\right)} \geq -\frac{1}{c} - 1 \Leftrightarrow c + c\sqrt{1 + \frac{1}{c}\left(\frac{1}{c} + 2\Re(a)\right)} \leq -1,$$

or equivalently $|x_1| \geq 1$. The case $c > 0$ can be treated in a similar way, and the proof follows.

□

From Theorem 2.6, one sees that for each $z_\alpha \in \mathbb{T} \backslash \{\pm 1\}$, a unique n-point symmetric Szegő-Lobatto rule can be constructed with prescribed nodes at z_α and $\overline{z}_\alpha$. To fix ideas, suppose n even, say $n = 2m$ with $m > 1$ and denote by $\tilde{I}_{2m}^\omega(g)$ the corresponding rule. From (2.7) the nodes of $\tilde{I}_{2m}^\omega(g)$ are the zeros of $\tilde{B}_{2m}(z, \tilde{\tau}_{2m})$ with $\tilde{\tau}_{2m} \in \{\pm 1\}$ (thus, a polynomial with real coefficients). Suppose that $\tilde{\tau}_{2m} = 1$ so that $\tilde{B}_{2m}(\pm 1, \tilde{\tau}_{2m}) = 2\tilde{\rho}_{2m-1}(\pm 1) \neq 0$ and hence the zeros of $\tilde{B}_{2m}(z, \tilde{\tau}_{2m})$ appear in complex conjugate pairs on $\mathbb{T}$. Then, $\tilde{I}_{2m}^\omega(g)$ has the form $\tilde{I}_{2m}^\omega(g) = A[g(z_\alpha) + g(\overline{z}_\alpha)] + \sum_{j=1}^{m-1} \lambda_j [g(z_j) + g(\overline{z}_j)]$, with $A > 0$ and $\lambda_j > 0$, $j = 1, \ldots, m$. Again, it is clearly seen that computation reduces by one half.

§3. Construction of Gauss-type formulas with preassigned nodes

Having summarized the most relevant properties of Szegő-type formulas in the previous sections, we will now see how these results can be used in order to characterize certain Gauss-type formulas associated with a weight function σ on $[-1,1]$. Thus, we return to the approximate calculation of the integral $I_\sigma(f) = \int_{-1}^{+1} f(x)\sigma(x)dx$. For this purpose, it should be recalled that σ gives rise to a symmetric 2π-periodic function ω by setting $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$. Furthermore, it holds that

$$I_\sigma(f) = \frac{1}{2}\int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta = \frac{1}{2}I_\omega(g), \quad \text{with} \quad g(e^{i\theta}) = f(\cos\theta) = f\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right).$$

A connection between quadrature formulas for ω and σ on $[-\pi,\pi]$ and $[-1,1]$ respectively is shown in the following (a proof is implicitly contained in [5] and also in [10]):

Proposition 3.1. Take $r, s \in \{0,1\}$ and consider $n-r-s$ distinct nodes $\{x_j^{r,s}\}_{j=1}^{n-r-s}$ on $(-1,1)$ along with the n real numbers $A_+^{r,s}$, $A_-^{r,s}$ and $\{A_j^{r,s}\}_{j=1}^{n-r-s}$. Set $x_j^{r,s} = \cos\theta_j^{r,s}$, $\theta_j^{r,s} \in (0,\pi)$ and define $z_j^{r,s} = e^{i\theta_j^{r,s}}$, $z_{n-r-s+j}^{r,s} = \overline{z_j^{r,s}}$ and $\lambda_j^{r,s} = \lambda_{n+j-r-s}^{r,s} = A_j^{r,s}$, $1 \le j \le n-r-s$. Then, the following statements are equivalent:

1. $I^{\sigma}_{n;(r,s)}(f) = rA_+^{r,s} f(1) + sA_-^{r,s} f(-1) + \sum_{j=1}^{n-r-s} A_j^{r,s} f(x_j^{r,s}) = I_\sigma(f)$, for all $f \in \mathcal{P}_N$.

2. $I^{\omega}_{2n-r-s}(g) = 2[rA_+^{r,s} g(1) + sA_-^{r,s} g(-1)] + \sum_{j=1}^{2(n-r-s)} \lambda_j^{r,s} g(z_j^{r,s}) = I_\omega(g)$, for all $g \in \Lambda_{-N,N}$.

Proposition 3.1 represents the key to perform the construction of the quadrature formula (1.5). Indeed, once $x_\alpha \in (-1,1)$ is fixed, consider the points z_α and $\overline{z}_\alpha$ on $\mathbb{T}\backslash\{\pm 1\}$ such that $\Re(z_\alpha) = x_\alpha$ and the $2n$-point symmetric Szegő-Lobatto rule $\tilde{I}^{\omega}_{2n}(g)$ for $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ with prescribed nodes z_α and $\overline{z}_\alpha$. As already seen this rule has the form,

$$\tilde{I}^{\omega}_{2n}(g) = A_\alpha[g(z_\alpha) + g(\overline{z_\alpha})] + \sum_{j=1}^{2(n-1)} \lambda_j g(z_j) = I_\omega(g), \quad \text{for all } g \in \Lambda_{-(n-2),n-2},$$

where A_α and λ_j, $j = 1, \dots, 2n-2$ are positive, the nodes $\{z_j\}_{j=1}^{2(n-1)}$ are real (± 1) or appear in complex conjugate pairs and the weights corresponding to two conjugate nodes are the same. Furthermore, the nodes are the zeros of

$$\tilde{B}_{2n}(z, \tilde{\tau}_{2n}) = z\tilde{\rho}_{2n-1}(z) + \tilde{\tau}_{2n}\tilde{\rho}^*_{2n-1}(z) \quad \text{with} \quad \tilde{\rho}_{2n-1}(z) = z\rho_{2n-2}(z) + \tilde{\delta}_{2n-1}\rho^*_{2n-2}(z), \tag{3.1}$$

$\rho_{2n-2}(z)$ being the monic Szegő polynomial of degree $2n-2$ for ω, $\tilde{\delta}_{2n-1} \in (-1,1)$ and $\tilde{\tau}_{2n} \in \{\pm 1\}$. Hence, there are two possibilities.

- $\underline{\tilde{\tau}_{2n} = 1.}$ From (3.1) one sees that $\tilde{B}_{2n}(\pm 1, 1) \neq 0$. Hence, its $2n$ zeros appear in complex conjugate pairs on $\mathbb{T}$. Thus, from Proposition 3.1 with $N = 2n-2$ one sees that a quadrature formula satisfying (1.5) can be constructed as $r = s = 0$. Indeed, setting $x_\alpha = \Re(z_\alpha)$, $A_\alpha^{0,0} = A_\alpha$, $x_j^{0,0} = \Re(z_j)$ and $A_j^{0,0} = \lambda_j$, $j = 1, \dots, n-1$ it follows that,

$$I_n^{0,0}(f) = A_\alpha^{0,0} f(x_\alpha) + \sum_{j=1}^{n-1} A_j^{0,0} f(x_j^{0,0}) = I_\sigma(f), \quad \text{for all } f \in \mathcal{P}_{2n-2}. \tag{3.2}$$

  Furthermore, the weights in (3.2) are all positive.

- $\underline{\tilde{\tau}_{2n} = -1.}$ From (3.1), $\{\pm 1\}$ are zeros of $\tilde{B}_{2n}(z,-1)$ and the remaining ones appear on $\mathbb{T}$ in complex conjugate pairs. The corresponding $2n$-point symmetric Szegő-Lobatto formula $\tilde{I}^{\omega}_{2n}(f)$ can be expressed as

$$\begin{aligned}\tilde{I}^{\omega}_{2n}(g) &= A_\alpha[g(z_\alpha) + g(\overline{z}_\alpha)] + A_+ g(1) + A_- g(-1) + \textstyle\sum_{j=1}^{n-2} \lambda_j [g(z_j) + g(\overline{z}_j)] \\ &= I_\omega(g), \quad \text{for all } g \in \Lambda_{-(2n-2),2n-2}.\end{aligned}$$

  Again by Proposition 3.1, but now replacing n by $n+1$, we have the solution to (1.5) as $r = s = 1$. Indeed, setting $x_\alpha = \Re(z_\alpha)$, $A_\alpha^{1,1} = A_\alpha$, $A_+^{1,1} = \frac{A_+}{2}$, $A_-^{1,1} = \frac{A_-}{2}$, $x_j^{1,1} = \Re(z_j)$ and $A_j^{1,1} = \lambda_j$, $j = 1, \dots, n-2$ it follows that,

$$\begin{aligned}I_{n+1}^{1,1}(f) &= A_+^{1,1} f(1) + A_-^{1,1} f(-1) + A_\alpha^{1,1} f(x_\alpha) + \textstyle\sum_{j=1}^{n-2} A_j^{1,1} f(x_j^{1,1}) \\ &= I_\sigma(f), \quad \text{for all } f \in \mathcal{P}_{2n-2}.\end{aligned}$$

On the other hand, consider the $(2n-1)$-point symmetric Szegő-Lobatto rule $\tilde{I}^{\omega}_{2n-1}(g)$ for $I_\omega(g)$ with prescribed nodes z_α and $\overline{z}_\alpha$. Now the nodes of this rule are the zeros of $\tilde{B}_{2n-1}(z, \tilde{\tau}_{2n-1}) = z\tilde{\rho}_{2n-2}(z) + \tilde{\tau}_{2n-1}\tilde{\rho}^*_{2n-1}(z)$, where $\tilde{\rho}_{2n-2}(z) = z\rho_{2n-3}(z) + \tilde{\delta}_{2n-2}\rho^*_{2n-3}(z)$, $\tilde{\delta}_{2n-2} \in (-1,1)$, $\tilde{\tau}_{2n-1} \in \{\pm 1\}$ and $\rho_{2n-3}(z)$ being the $(2n-3)$-th monic Szegő polynomial for ω. As before, we will analyze the two possibilities for $\tilde{\tau}_{2n-1}$.

- $\underline{\tilde{\tau}_{2n-1} = 1}$. Since $\tilde{\rho}_{2n-2}(\pm 1) = \tilde{\rho}^*_{2n-2}(\pm 1)$ it follows that $z = -1$ is the unique real zero of $\tilde{B}_{2n-1}(z, 1)$ meanwhile the other $2n-2$ remaining ones lie on $\mathbb{T}$ in complex conjugate pairs yielding,

$$\begin{aligned} \tilde{I}^{\omega}_{2n-1}(g) &= A_- g(-1) + A_\alpha[g(z_\alpha) + g(\overline{z}_\alpha)] + \textstyle\sum_{j=1}^{n-2} \lambda_j [g(z_j) + g(\overline{z}_j)] \\ &= I_\omega(g), \quad \text{for all } g \in \Lambda_{-(2n-3),2n-3}. \end{aligned}$$

Again, when taking $N = 2n-3$, Proposition 3.1 produces the solution to (1.5) as $r = 0$ and $s = 1$. Indeed, setting $A^{0,1}_- = \frac{A_-}{2}$, $A^{0,1}_\alpha = A_\alpha$, $x^{0,1}_j = \Re(z_j)$ and $A^{0,1}_j = \lambda_j$, $j = 1, \ldots, n-2$. Then,

$$\begin{aligned} I^{0,1}_n(f) &= A^{0,1}_- f(-1) + A^{0,1}_\alpha f(x_\alpha) + \textstyle\sum_{j=1}^{n-2} A^{0,1}_j f(x^{0,1}_j) \\ &= I_\sigma(f), \quad \text{for all } f \in \mathcal{P}_{2n-3}. \end{aligned}$$

- $\underline{\tilde{\tau}_{2n-1} = -1}$. Now, $\tilde{B}_{2n-1}(z, -1)$ has a unique real zero at $z = 1$ and the remaining ones on $\mathbb{T}$ in complex conjugate pairs. Hence, the corresponding $(2n-1)$-th Szegő-Lobatto formula can be expressed as:

$$\begin{aligned} \tilde{I}^{\omega}_{2n-1}(g) &= A_+ g(1) + A_\alpha[g(z_\alpha) + g(\overline{z}_\alpha)] + \textstyle\sum_{j=1}^{n-2} \lambda_j [g(z_j) + g(\overline{z}_j)] \\ &= I_\omega(g), \quad \text{for all } g \in \Lambda_{(2n-3),2n-3}. \end{aligned}$$

From Proposition 3.1 with $N = 2n-3$, the solution to (1.5) as $r = 1$ and $s = 0$, directly follows. Indeed, set $A^{1,0}_+ = \frac{A_+}{2}$, $A^{1,0}_\alpha = A_\alpha$, $x^{1,0}_j = \Re(z_j)$ and $A^{1,0}_j = \lambda_j$, $j = 1, \ldots, n-2$. Then,

$$\begin{aligned} I^{1,0}_n(f) &= A^{1,0}_+ f(1) + A^{1,0}_\alpha f(x_\alpha) + \textstyle\sum_{j=1}^{n-2} A^{1,0}_j f(x^{1,0}_j) \\ &= I_\sigma(f), \quad \text{for all } f \in \mathcal{P}_{2n-3}. \end{aligned}$$

From the above analysis, Proposition 3.1, Theorem 2.5 along with the unicity of the quadrature formula $I^{r,s}_n(f)$ satisfying (1.5), the following Theorem can be proved,

Theorem 3.2. Given r and s in $\{0, 1\}$ and $x_\alpha \in (-1, 1)$, take $z_\alpha \in \mathbb{T}$ such that $x_\alpha = \Re(z_\alpha)$. Let σ be a weight function on $[-1, 1]$ and define $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$, $\theta \in [-\pi, \pi]$. For $n > 1 + r + s$ consider the $(2n-r-s)$-point Szegő-Lobatto symmetric quadrature rule to $I_\omega(g)$ with prescribed nodes at z_α and $\overline{z}_\alpha$. Let $\tilde{\delta}_{2n-(r+s+1)} \in (-1, 1)$ and $\tilde{\tau}_{2n-(r+s)} \in \{\pm 1\}$ be the parameters characterizing this rule. Then, the quadrature rule $I^{r,s}_n(f)$ constructed above coincide with (1.5), if and only if, $\tilde{\tau}_{2n-(r+s)} = (-1)^r$.

Remark 3.3. Because of the positivity of the weights in a Szegő-Lobatto formula, the weights in any formula $I^{r,s}_n(f)$ satisfying (1.5) are positive.

Now, it remains to show how the conditions in Theorem 3.2 can be rewritten in terms of the information we are initially handling i.e., information on the weight function σ together with the fixed node $x_\alpha \in (-1, 1)$. For our purposes and for the sake of simplicity, we will restrict ourselves to the n-point quadrature (1.5) as $r = s = 0$ so that the remaining other cases can be considered in a similar way. Thus, take,

$$I^{0,0}_n(f) = I^\sigma_n(f) = A_\alpha f(x_\alpha) + \sum_{j=1}^{n-1} A_j f(x_j), \tag{3.3}$$

where for simplicity in the notation we have suppressed the super-index $(0, 0)$. Set $R_n(x) = (x - x_\alpha)\prod_{j=1}^{n-1}(x - x_j)$, and let $\{p_k\}_{k=0}^\infty$ denote any sequence of orthogonal polynomials for σ. Then by Theorem 1.2, $\tilde{C}_n R_n(x) = p_n(x) - C_n p_{n-1}(x)$ with $\tilde{C}_n \neq 0$. Since $R_n(x_\alpha) = 0$, then $p_{n-1}(x_\alpha) \neq 0$ because otherwise, p_n and p_{n-1} would have a common zero at x_α. Thus, we can write $C_n = f_n(x_\alpha)$ where $f_n(x) = \frac{p_n(x)}{p_{n-1}(x)}$ for all $n \geq 1$. It is known (see e.g. [35]) that the polynomial $R_n(x)$ has at least $n-1$ distinct zeros in $(-1, 1)$. However, one zero might lie outside this interval. Thus, our aim is to give conditions on x_α so that all the zeros of $R_n(x)$ are distinct and lie in $(-1, 1)$. Therefore, we will make use of the known relation between sequences of orthogonal polynomials associated with

σ and ω respectively. Thus, if p_n denotes an orthogonal polynomial for σ of degree n (conveniently normalized) and ρ_{2n} the $2n$-th monic Szegő polynomial for ω, it holds that (see [38]),

$$p_n(x) = \frac{\rho_{2n}(z) + \rho_{2n}^*(z)}{z^n} = \frac{B_{2n}(z,1)}{z^n}, \quad z = e^{i\theta}, \quad x = \cos\theta, \tag{3.4}$$

with $B_{2n}(z) \in \mathcal{P}_{2n}$. Hence, one can write (up to a multiplicative factor) $R_n(x) = z^{-n}[B_{2n}(z) - zB_{2n-2}(z)]$. Moreover, from the Szegő recurrence formulas (2.5) it follows that,

$$\begin{aligned} R_n(x) &= \tfrac{1+\delta_{2n}}{z^n}\left[z(z+\delta_{2n-1} - \tfrac{C_n}{1+\delta_{2n}})\rho_{2n-2}(z) + (1 + z(\delta_{2n-1} - \tfrac{C_n}{1+\delta_{2n}}))\rho_{2n-2}^*(z)\right] \\ &= \tfrac{1+\delta_{2n}}{z^n} T(z), \end{aligned} \tag{3.5}$$

with $x = \cos\theta$, $z = e^{i\theta}$ and $T(z)$ being a monic polynomial of degree $2n$. Thus, if we are assuming that $R_n(x)$ has n distinct zeros in $(-1,1)$ and denoted by $x_\alpha, x_1, \ldots, x_{n-1}$ such that $x_\alpha = \cos\alpha$ and $x_j = \cos\theta_j$, $j = 1,\ldots,n-1$ with $\alpha, \theta_j \in (0,\pi)$, then the polynomial $T(z)$ has $2n$ distinct zeros on $\mathbb{T}$ appearing in complex conjugate pairs: $\{z_\alpha, \overline{z}_\alpha\} \cup \{z_j, \overline{z}_j\}_{j=1}^{n-1}$, where $z_\alpha = e^{i\alpha}$ and $z_j = e^{i\theta_j}$, $j = 1,\ldots,n-1$. By Proposition 3.1, one sees that the zeros of $T(z)$ are the nodes of the $2n$-point symmetric Szegő-Lobatto rule for $I_\omega(g)$ with prescribed nodes at z_α and $\overline{z}_\alpha$. Therefore,

$$T(z) = z\tilde{\rho}_{2n-1}(z) + \tilde{\tau}_{2n}\tilde{\rho}_{2n-1}^*(z), \quad \text{with} \quad \tilde{\rho}_{2n-1}(z) = z\rho_{2n-2}(z) + \tilde{\delta}_{2n-1}\rho_{2n-2}^*(z). \tag{3.6}$$

A comparison between the expressions (3.5) and (3.6) easily yields $\tilde{\tau}_{2n} = 1$ (as expected) and $\tilde{\delta}_{2n-1} = \delta_{2n-1} - \frac{C_n}{1+\delta_{2n}}$. Thus, the existence of the quadrature rule $I_n^\sigma(f)$ given by (3.3) to be exact in $\mathcal{P}_{2n-2}$ is equivalent to $\tilde{\delta}_{2n-1} \in (-1,1)$ yielding,

$$|\tilde{\delta}_{2n-1}| < 1 \Leftrightarrow (1+\delta_{2n})(\delta_{2n-1} - 1) < f_n(x_\alpha) < (1+\delta_{2n})(1+\delta_{2n-1}) \tag{3.7}$$

with $f_n(x_\alpha) = \frac{p_n(x_\alpha)}{p_{n-1}(x_\alpha)}$. On the other hand, by (3.4) and since the Szegő polynomials have real coefficients (because of the symmetry of ω) one has

$$f_n(1) = \frac{\rho_{2n}(1)}{\rho_{2n-2}(1)} \quad \text{and} \quad f_n(-1) = -\frac{\rho_{2n}(-1)}{\rho_{2n-2}(-1)}. \tag{3.8}$$

In [36] we can find the following relation between the L_2^ω-norm of $\rho_n(z)$, $\rho_n(\pm1)$ and the corresponding Verblunsky coefficients:

$$\rho_n(1) = \|\rho_n\|_\omega \prod_{j=1}^n \sqrt{\frac{1+\delta_j}{1-\delta_j}} \quad \text{and} \quad \rho_n(-1) = (-1)^n\|\rho_n\|_\omega \prod_{j=1}^n \sqrt{\frac{1+(-1)^{j+1}\delta_j}{1-(-1)^{j+1}\delta_j}},$$

where $\|\rho_n\|_\omega^2 = \int_{-\pi}^{\pi} |\rho_n(e^{i\theta})|^2\omega(\theta)d\theta = \prod_{j=1}^n(1-|\delta_j|^2)$. From (3.8), these relations allow us to write,

$$f_n(1) = (1+\delta_{2n})(1+\delta_{2n-1}) \ \text{ and } f_n(-1) = -(1-\delta_{2n})(1+\delta_{2n-1}).$$

Finally, by Theorem 3.2 and (3.7) one concludes that the existence of the n-point quadrature rule (3.3) with a prescribed node at $x_\alpha \in (-1,1)$ is equivalent to the condition $f_n(-1) < f_n(x_\alpha) < f_n(1)$. This result was also recently deduced in [4] making use of the theory of orthogonal polynomials with respect to the weight function σ in $[-1,1]$. Our approach by passing to the unit circle and taking advantage of the symmetric Szegő-Lobatto formulas results in much more straightforward and simpler.

§4. Computation

In this section we will show how the n-point quadrature formula $I_n^{r,s}(f)$ given by (1.5) with a prescribed node at $x_\alpha \in (-1,1)$ and possible nodes at ±1 can be efficiently computed by solving an eigenvalue problem of dimension n, involving certain tri-diagonal matrices. From Proposition 3.1 and Theorem 3.2, one sees that the computation of $I_n^{r,s}(f)$ associated with the integral $I_\sigma(f) = \int_{-1}^{+1} f(x)\sigma(x)dx$, σ being a weight function on $[-1,1]$ is equivalent to the computation of the $(2n-r-s)$-point Szegő-Lobatto formula for $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$, where $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$, with prescribed nodes z_α and $\overline{z}_\alpha$ on $\mathbb{T}$ such that $x_\alpha = \Re(z_\alpha)$. If we denote by $\tilde{I}_{2n-r-s}^\omega(g)$ this formula, it holds that

$$\tilde{I}_{2n-r-s}^\omega(L) = I_\omega(L), \quad \text{for all } L \in \Lambda_{-[2(n-1)-r-s],2(n-1)-r-s}.$$

Furthermore, there exists a new symmetric weight function $\tilde{\omega}(\theta)$ such that,

$$\tilde{I}^{\omega}_{2n-r-s}(g) = I_{\tilde{\omega}}(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\tilde{\omega}(\theta)d\theta, \quad \text{for all } g \in \Lambda_{-(2n-r-s-1),2n-r-s-1}$$

i.e., $I^{\omega}_{2n-r-s}(g)$ is actually a symmetric Szegő rule for $I_{\tilde{\omega}}(g)$. At the same time, by Proposition 3.1 one can easily deduce that $I^{r,s}_n(f)$ represents a Gauss-type (Gauss, Gauss-Radau or Gauss-Lobatto) formula for $I_{\tilde{\sigma}}(f)$, $\tilde{\sigma}$ being a weight function on $[-1,1]$ such that $\tilde{\omega}(\theta) = \tilde{\sigma}(\cos\theta)|\sin\theta|$. If we denote by $\{\delta_k\}_{k=0}^{\infty}$ and by $\{\tilde{\delta}_k\}_{k=0}^{\infty}$ ($\delta_0 = \tilde{\delta}_0 = 1$) the Verblunsky coefficients for ω and $\tilde{\omega}$ respectively, it holds that $\tilde{\delta}_k = \delta_k$, $k = 0,1,\ldots,n-2$. Furthermore, $\tilde{\delta}_{n-1}$ depends on $\{\delta_k\}_{k=0}^{n-2}$ and it can be easily computed (see Theorem 2.6).

In short, the computation of $I^{r,s}_n(f)$, if it exists, reduces to the computation of a certain n-point Gauss-type formula for the new weight function $\tilde{\sigma}$. Thus, if we assume that we have at our disposal the Jacobi parameters $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ of σ, the first step will be to compute for each n the Jacobi parameters $\{\tilde{a}_k\}_{k=1}^{n-1}$ and $\{\tilde{b}_k\}_{k=0}^{n-1}$ associated with $\tilde{\sigma}$. In this regard, we need the following Geronimus relations connecting the Verblunsky coefficients and the Jacobi parameters for ω and σ respectively (see [19]).

Theorem 4.1. Let ω be a symmetric weight function on $[-\pi,\pi]$ satisfying $\mu_0 = \int_{-\pi}^{\pi}\omega(\theta)d\theta = 1$ and σ the weight function on $[-1,1]$ related to ω by $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$. Let $\{\delta_k\}_{k=0}^{\infty}$ be the sequence of Verblunsky parameters for ω and $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be the coefficients of the Jacobi matrix for σ. Then, the following holds:

$$2a_n = \sqrt{(1-\delta_{2n})(1-\delta_{2n-1}^2)(1+\delta_{2n-2})}, \;\; n \geq 1, \quad 2b_n = \delta_{2n-1}(1-\delta_{2n}) - \delta_{2n+1}(1+\delta_{2n}), \;\; n \geq 0. \tag{4.1}$$

Relations (4.1) allow us to easily compute the Jacobi coefficients in terms of the Verblunsky parameters. However our initial information is concerning σ and hence its Jacobi coefficients. For this purpose, we proceed as in [18]. Define the sequence $\{u_k\}_{k=0}^{\infty}$ as,

$$u_k = \frac{1}{2}(1-\delta_k)(1+\delta_{k-1}), \tag{4.2}$$

so that from (4.1) one obtains $a_k^2 = u_{2k}u_{2k-1}$, $k \geq 1$ and $b_k + 1 = u_{2k} + u_{2k+1}$, $k \geq 0$. From here the unique LU factorization follows: $\mathcal{J} + I = \mathcal{L}\mathcal{U}$, where I is the identity matrix, $\mathcal{L}$ and $\mathcal{U}$ lower and upper bi-diagonal matrices respectively given by,

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ u_2 & 1 & 0 & 0 & \cdots \\ 0 & u_4 & 1 & 0 & \cdots \\ 0 & 0 & u_6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} u_1 & 1 & 0 & 0 & \cdots \\ 0 & u_3 & 1 & 0 & \cdots \\ 0 & 0 & u_5 & 1 & \cdots \\ 0 & 0 & 0 & u_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\mathcal{J} = \begin{pmatrix} b_0 & 1 & 0 & 0 & \cdots \\ a_1^2 & b_1 & 1 & 0 & \cdots \\ 0 & a_2^2 & b_2 & 1 & \cdots \\ 0 & 0 & a_3^2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{(monic Jacobi matrix).}$$

Thus, once we have computed the sequence $\{u_k\}_{k=1}^{\infty}$ from the LU decomposition of the matrix $\mathcal{J}+I$, by (4.2) one recursively obtains the parameters $\{\delta_k\}_{k=0}^{\infty}$,

$$\delta_k = 1 - \frac{2u_k}{1+\delta_{k-1}}, \; k \geq 1, \tag{4.3}$$

or equivalently,

$$\delta_k = 1 - \cfrac{2u_k}{2 - \cfrac{2u_{k-1}}{2 - \cfrac{2u_{k-2}}{\ddots \, 2 - \cfrac{}{2 - \cfrac{2u_2}{2-u_1}}}}}. \tag{4.4}$$

The above considerations can be summarized as follows: suppose σ a weight function on $[-1,1]$ and let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ be its corresponding Jacobi parameters. For fixed $x_\alpha \in (-1,1)$, we will be concerned with the computation of the nodes and weights of the rule $I_n^{r,s}(f)$ as given by (1.5). To fix ideas, we will restrict ourselves to the case $r = s = 0$ i.e.,

$$I_n^{0,0}(f) = I_n^{\sigma}(f) = A_\alpha f(x_\alpha) + \sum_{j=1}^{n-1} A_j f(x_j), \tag{4.5}$$

with $\{x_j\}_{j=1}^{n-1} \subset (-1,1)$, $x_j \neq x_k$ if $j \neq k$ for all $1 \le j,k \le n-1$, $x_j \neq x_\alpha$ for all $j = 1,\ldots,n-1$ and such that $I_n^{\sigma}(P) = I_\sigma(P)$ for all $P \in \mathcal{P}_{2n-2}$. Setting as usual, $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ and $z_\alpha \in \mathbb{T}$ such that $x_\alpha = \Re(z_\alpha)$, from Theorem 3.2 we see that our problem reduces to study the $2n$-point symmetric Szegő-Lobatto rule to $I_\omega(g)$ with prescribed nodes z_α and $\overline{z}_\alpha$ but under the restriction that ± 1 are not nodes or equivalently that the parameter $\tilde{\tau}_{2n}$ in Theorem 3.2 is equal to 1.

By denoting $\{\delta_k\}_{k=0}^{\infty}$, the Verblunsky parameters for ω and taking into account that for $n > 1$ we only need $\delta_0, \delta_1, \ldots, \delta_{2n-2}$, computation can be organized in the following steps:

Step 1 Proceed with the LU factorization of the matrix $\mathcal{J}_n + I_n$ where for $n \ge 1$, $\mathcal{J}_n$ represent the principal submatrix of order n of $\mathcal{J}$ and I_n the identity matrix of order n.

Step 2 The above factorization produces the numbers $\{u_k\}_{k=0}^{2n-1}$, $(u_0 = 0)$. Thus, by (4.3) or (4.4) we can easily compute recursively the parameters $\delta_0, \delta_1, \ldots, \delta_{2n-2}$ along with the two parameters characterizing the $2n$-point Szegő-Lobatto formula $\tilde{\delta}_{2n-1} \in (-1,1)$ and $\tilde{\tau}_{2n} \in \{\pm 1\}$ (see Theorem 3.2). Since we are dealing with the case $r = s = 0$, clearly $\tilde{\tau}_{2n} = 1$ implies existence of $I_n^{\sigma}(f)$ and $\tilde{\tau}_{2n} = -1$ implies non existence.

Step 3 The numbers $\delta_0, \delta_1, \ldots, \delta_{2n-2}$ and $\tilde{\delta}_{2n-1}$, as already said, represent the first $2n$ Verblunsky parameters of a new symmetric weight function $\tilde{\omega}(\theta)$. Assume $\tilde{\omega}(\theta) = \tilde{\sigma}(\cos\theta)|\sin\theta|$ and let $\{\tilde{a}_k\}_{k=1}^{\infty}$ and $\{\tilde{b}_k\}_{k=1}^{\infty}$ be the Jacobi coefficients for $\tilde{\sigma}(x)$. Then from the Geronimus relations (4.1) one has,

$$\tilde{a}_k = a_k,\ 1 \le k \le n-1, \quad \tilde{b}_k = b_k,\ 0 \le k \le n-2, \quad 2\tilde{b}_{n-1} = \delta_{2n-3}[1 - \delta_{2n-2}] - \tilde{\delta}_{2n-1}[1 + \delta_{2n-2}].$$

Step 4 Consider the matrix of order n

$$\begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots & 0 \\ a_1 & b_1 & a_2 & 0 & \cdots & 0 \\ 0 & a_2 & b_2 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & \tilde{b}_{n-1} \end{pmatrix} \tag{4.6}$$

which represents the n-th truncation of the Jacobi matrix associated with $\tilde{\sigma}(x)$. Hence, its eigenvalues are the nodes of $I_n^{\sigma}(f)$ in (4.5) and the square of the first component of the eigenvector of unit length corresponding to the eigenvalue (node) x_j yields the weight A_j.

On the other hand, since we know x_α is an eigenvalue of (4.6) we could use a deflation method so that we should only compute $x_1, \ldots, x_{n-1}$ and $A_1, \ldots, A_{n-1}$. Observe that,

$$A_\alpha = c_0 - \sum_{j=1}^{n-1} A_j, \text{ with } c_0 = \int_{-1}^{+1} \sigma(x)dx.$$

Remark 4.2. The analysis of the other quadrature rules $I_n^{r,s}(f)$ in (1.5) with $r, s \in \{0, 1\}$ and $(r, s) \neq (0, 0)$ leads clearly to the computation of the n-point Gauss-Radau and Gauss-Lobatto formulas for $I_{\tilde{\sigma}}(f) = \int_{-1}^{+1} f(x)\tilde{\sigma}(x)dx$.

§5. Numerical examples involving Bernstein-Szegő polynomials

The weight functions on the interval $[a, b]$ of the form $\sigma(x) = (x-a)^\alpha(b-x)^\beta$ with $\alpha, \beta > -1$ used to be called of Jacobi-type since they give rise to the family of orthogonal polynomials known as Jacobi polynomials. When restricting ourselves to the interval $[-1, 1]$ and taking $\alpha, \beta \in \{\pm\frac{1}{2}\}$, the Chebyshev-type weight functions appear and give rise to the family of Chebyshev polynomials of different orders so that the corresponding Gaussian formulas are the only ones whose coefficients or weights are explicitly known.

On the other hand, when one desires to approximate a weighted integral $I_\sigma(f) = \int_a^b f(x)\sigma(x)dx$ where the integrand f exhibits singularities near $[a, b]$ but not on this interval, a common technique consists of collecting all the singularities in a new weight function $\tilde{\sigma}$ and writting $I_\sigma(f) = \int_a^b g(x)\tilde{\sigma}(x)dx$, g being a function that is smooth enough. When dealing with polar singularities one could write $f = g/P$ so that $I_\sigma(f) = \int_a^b g(x)\frac{\sigma(x)}{P(x)}dx$, with P a polynomial such that $P(x) > 0$ on $[a, b]$. Thus, the theory of orthogonal polynomials and related topics with respect to rational modifications of a measure arises (see e.g. [1], [21] and [32]). When $[a, b] = [-1, 1]$ and σ is of Chebyshev-type and one considers $\tilde{\sigma} = \sigma/P$, with $P(x) > 0$ on $[-1, 1]$, then the family of orthogonal polynomials on the unit circle for $\tilde{\omega}(\theta) = \tilde{\sigma}(\cos\theta)|\sin\theta| = \frac{\omega(\theta)}{|h(e^{i\theta})|^2}$ with $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$ and $h(z)$ a polynomial of the same degree as $P(x)$, are the so-called Bernstein-Szegő polynomials ([38]). In this respect, several numerical experiments involving Bernstein-Szegő polynomials will be done in order to illustrate the computation of the Gauss-type quadratures for the weight function of the form $\sigma(x) = \frac{(1-x)^\alpha(1+x)^\beta}{P(x)}$ with $\alpha, \beta \in \{\pm\frac{1}{2}\}$. To fix ideas, we will concentrate on the particular case $\alpha = \beta = -1/2$ (Chebyshev-type of the first kind) and $P(x) = (1-\gamma x)^m$ with $\gamma \in (-1, 1)$ i.e., $\sigma(x) = \frac{1}{(1-\gamma x)^m\sqrt{1-x^2}}$. For further details concerning Gaussian formulas for this type of weight functions, see [12] and [2]. In this case, we have (apart from a multiplicative factor) $\omega(\theta) = \sigma(\cos\theta)|\sin\theta| = |z - \tilde{\gamma}|^{-2m}$ with $z = e^{i\theta}$ and $\tilde{\gamma} \in (-1, 1)$ (see [2]) so that the monic Bernstein-Szegő polynomials are explicitly given by $\rho_n(z) = z^{n-m}(z-\tilde{\gamma})^m$ for all $n \geq m$ and consequently the Verblunsky parameters δ_n are zero for $n > m$.

Now, making use of Theorem 3.2 along with (2.9) and (2.10), the following can be easily proved,

Proposition 5.1. Consider the weight function $\sigma(x) = \sigma(x, \gamma, m) = \frac{1}{(1-\gamma x)^m\sqrt{1-x^2}}$ with m a nonnegative integer and $\gamma \in (-1, 1)$ and fix $x_\alpha \in (-1, 1)$. Then, the corresponding quadrature rule $I_n^{r,s}(f)$ given by (1.5) with $2n \geq m + r + s$ will exist, if and only if,

$$(-1)^r \Im\left(z_\alpha^{2(m-n+(r+s))}\left(\frac{\overline{z}_\alpha - \tilde{\gamma}}{z_\alpha - \tilde{\gamma}}\right)^m\right) \leq 0, \tag{5.1}$$

where $z_\alpha = x_\alpha + i\sqrt{1-x_\alpha^2}$ and $\tilde{\gamma} = 2(\frac{1}{\gamma} + \gamma)^{-1}$.

Remark 5.2. When taking above $m = 0$, then $\sigma(x) = \frac{1}{\sqrt{1-x^2}}$ (Chebyshev weight function of the first kind) so that condition (5.1) now becomes $(-1)^r \sin(2(n-r-s)\alpha) \geq 0$, by assuming $x_\alpha = \cos\alpha$ with $\alpha \in (0, \pi)$.

Now, we will check the existence of the quadrature rule $I_n^{0,0}(f)$ for $n = 5$ associated with $\sigma(x, \gamma, m)$ for different values of x_α, γ and m. Recall that existence is equivalent to the fact the parameters $\tilde{\tau}_{2n}$ is equal to one. As an illustration we have also computed the "modified Verblunsky parameter" $\tilde{\delta}_{2n-1} \in (-1, 1)$. The results are shown on Tables 2, 3 and 4. Finally on Tables 5, 6 and 7 the nodes and weights of the quadrature rule $I_5^{0,0}(f)$ for $\sigma(x, \gamma, m)$ are displayed, when taking as above different values of the fixed node x_α and the parameters $m \geq 0$ and $\gamma \in (-1, 1)$. All the computations were done with MAPLE® 9.5[3].

[3]MAPLE is a registered trademark of Waterloo Maple, Inc.

γ	x_α	$\tilde{\delta}_{2n-1}$	$\tilde{\tau}_{2n}$
0.198	0.5403023059	0.6292798876	1
	−0.4161468365	0.6320230660	−1
	0.2836621855	0.2637597967	−1
	−0.1455000338	0.9476575187	1
0.8	0.5403023059	0.7016549910	−1
	−0.4161468365	0.9199641453	−1
	0.2836621855	−0.9990300645	1
	−0.1455000338	0.4674292023	−1
−0.975	0.5403023059	−0.1654997820	1
	−0.4161468365	0.3231354585	−1
	0.2836621855	0.4172583460	−1
	−0.1455000338	−0.0003754590	1

Table 2: Values of $\tilde{\delta}_{2n-1}$ and $\tilde{\tau}_{2n}$ in a 5 point Gauss-type quadrature formula with a prefixed node x_α for the weight function $\sigma(x, \gamma, 1) = (1 - \gamma x)^{-1}(1 - x^2)^{-1/2}$.

γ	x_α	$\tilde{\delta}_{2n-1}$	$\tilde{\tau}_{2n}$
0.198	0.5403023059	0.8796981024	1
	−0.4161468365	−0.7176506326	−1
	0.2836621855	−0.3967315251	−1
	−0.1455000338	0.8576175107	−1
0.8	0.5403023059	0.2487954650	−1
	−0.4161468365	−0.3573679249	1
	0.2836621855	−0.4104362350	1
	−0.1455000338	−0.0236154680	−1
−0.975	0.5403023059	−0.5574209200	1
	−0.4161468365	0.4703245040	1
	0.2836621855	0.8612859631	1
	−0.1455000338	−0.9686333180	−1

Table 3: Values of $\tilde{\delta}_{2n-1}$ and $\tilde{\tau}_{2n}$ in a 5 point Gauss-type quadrature formula with a prefixed node x_α for the weight function $\sigma(x, \gamma, 2) = (1 - \gamma x)^{-2}(1 - x^2)^{-1/2}$.

γ	x_α	$\tilde{\delta}_{2n-1}$	$\tilde{\tau}_{2n}$
0.198	0.5403023059	0.7720906475	−1
	−0.4161468365	−0.9675093827	1
	0.2836621855	−0.9555341247	−1
	−0.1455000338	0.3958561706	−1
0.8	0.5403023059	−0.2506629030	1
	−0.4161468365	0.7740235707	1
	0.2836621855	0.4519582710	−1
	−0.1455000338	−0.2694822410	1
−0.975	0.5403023059	0.3412054340	−1
	−0.4161468365	0.6308007416	−1
	0.2836621855	−0.6425455683	−1
	−0.1455000338	0.2181996360	1

Table 4: Values of $\tilde{\delta}_{2n-1}$ and $\tilde{\tau}_{2n}$ in a 5 point Gauss-type quadrature formula with a prefixed node x_α for the weight function $\sigma(x, \gamma, 5) = (1 - \gamma x)^{-5}(1 - x^2)^{-1/2}$.

Nodes	Weights
−0.97035702489884	0.13392360961570
−0.68189701062008	0.17294724994672
−0.09789314234364	0.20380393821627
0.54030230586814	0.23382327343744
0.94520492821188	0.25550192878387

Table 5: Nodes and weights of the quadrature rule for the weight function $\sigma(x,\gamma,1) = (1-\gamma x)^{-1}(1-x^2)^{-1/2}$, $\gamma = 0.198$ and $x_\alpha = 0.5403023059$.

Nodes	Weights
−0.92665900212908	0.01742444926955
−0.41614683654714	0.02856457507296
0.27257424007750	0.07461067432043
0.77100893618287	0.25265989346182
0.97790662465163	0.62674040787524

Table 6: Nodes and weights of the quadrature rule for the weight function $\sigma(x,\gamma,2) = (1-\gamma x)^{-2}(1-x^2)^{-1/2}$, $\gamma = 0.8$ and $x_\alpha = -0.4161468365$.

§6. Conclusions

As already said, the approximate calculation of weighted integrals over a finite interval $[a,b]$ with one or two prescribed nodes at the end points is obtained by the commonly named Gauss-Radau and Gauss-Lobatto quadrature formulas. These rules achieve the maximum domain of validity when the nodes are the zeros of orthogonal polynomials on the real line with respect to a certain weight function and such zeros can be efficiently obtained as eigenvalues of a tri-diagonal Jacobi matrix.

In the recent paper [4] some of the authors have characterized Gauss-Radau and Gauss-Lobatto quadrature formulas with one or two preassigned nodes inside the interval of integration by analyzing properties of the zeros of quasi-orthogonal polynomials on the real line. Such rules do not always exist.

In this paper we have used the relation between quadrature formulas for a weight function σ on $[-1,1]$ and a corresponding symmetric weight function ω on $\mathbb{T}$, given by $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$. Two advantages are delivered by this relation that form the main contributions of this paper. Firstly when dealing with the computation of a symmetric Szegő-type formula as an eigenvalue problem, we proved that it can be reduced in size via a Hessenberg matrix of dimension n to a Jacobi matrix of dimension $E[n]$, where $E[x]$ denotes the integer part of x. Secondly, in the analysis of the existence, unicity and characterization of a Gauss-type quadrature formula with a preassigned node in $(-1,1)$, we have presented an alternative and simpler approach by passing to the unit circle and using the characterization of Szegő-Lobatto rules stated in [26]. In this way we recover a part of the results recently obtained in [4]. This alternative method can not be applied to the problem when two distinct nodes are prefixed in $(-1,1)$ since we need in this case the characterization of a symmetric Szegő formula with four prefixed nodes (two complex conjugate pairs). More generally, it is still an open problem to study the existence, unicity and characterization of symmetric Szegő-type quadrature formulas for a symmetric weight on $\mathbb{T}$, with an arbitrary number of complex conjugate pairs of nodes prefixed in advance. Such characterization would give the link to study Gauss-type quadrature rules on the interval $[-1,1]$ with an arbitrary number of prefixed nodes given in advance. This more general problem will be considered in a forthcoming paper.

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Nodes	Weights
−0.99626590974071	0.82277669947417
−0.92371281332772	0.16466476654577
−0.50317756406768	0.01000588863206
0.28366218546323	0.00173973046429
0.90885111943214	0.00081291488371

Table 7: Nodes and weights of the quadrature rule for the weight function $\sigma(x, \gamma, 2) = (1 - \gamma x)^{-2}(1 - x^2)^{-1/2}$, $\gamma = -0.975$ and $x_\alpha = 0.2836621855$.

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